ASYMPTOTIC METHODS IN CONTACT PROBLEMS OF ELASTICITY THEORY

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Approximately in the thirties the demands of practice lead to the necessity for studying a number of contact problems of elasticity theory, which differed considerably in formulation from the classical problems of Boussinesq, Hertz and Sadovskii. The first investigations of such problems, namely, contact problems for an elastic strip, layer and cylinder, were performed in [1 — 5]. However, the explosive development of this problem, which we agree to call "nonclassical contact problems", is only observed at the beginning of the sixties, when the appropriate mathematical opportunities for effective solution of complex mixed problems of mathematical physics became manifest.

Recent experience has shown convincingly that the most effective apparatus for obtaining practically applicable approximate solutions of nonclassical contact problems are asymptotic methods. The first attempts in this area were in [6 and 7].

In the majority of cases nonclassical contact problems of elasticity theory result in the investigation of special types of integral equations of the first kind with symmetric, nonregular kernels of complicated structure. Hence, we deal below with effective asymptotic methods of solving such integral equations.

An important achievement of the mentioned methods is that they can be used in both the solution of plane and three-dimensional (axisymmetric or nonaxisymmetric) contact problems. Taking this into account, the crux of asymptotic methods which have been developed will be expounded below in sufficient detail by means of the example of plane contact problems, and the just mentioned application of these methods to three-dimensional problems.

It should be noted that the asymptotic methods elucidated here can be used successfully to solve not only contact problems of elasticity theory, but also an entire series of mixed problems of mathematical physics, namely, many mixed problems of hydro-aeromechanics.

1. General properties of the kernel and the solution of the fundamental integral equation of plane nonclassical contact problems. Let us consider an integral equation of the form

$$\int_{-1}^{1} \varphi(\xi) K\left(\frac{\xi - x}{\lambda}\right) d\xi = \pi f(x), \qquad |x| \leqslant 1 \tag{1.1}$$

where λ is some characteristic nondimensional parameter for any kind of contact problem, and $\lambda \in (0, \infty)$. The kernel of the integral equation is

$$K(t) = \int_0^\infty \frac{L(u)}{u} \cos ut \, du \tag{1.2}$$

where the function $L(z)z^{-1}$ will be even and meromorphic in the complex $z = u + iv$ plane (the ratio of two quasipolynomials of the same order), real and regular for $v = 0$ and $\lim L(z)z^{-1} = A > 0$ for $z \to 0$. As is known, such a function can be represented as

$$\frac{L(z)}{z} = \prod_{n=1}^{\infty} \frac{(z^2 + \delta_n^2)}{(z^2 + \gamma_n^2)}, \qquad \delta_n = -iz_n, \qquad \gamma_n = -i\zeta_n \tag{1.3}$$

Here z_n and ζ_n are the zeroes and poles in the $v > 0$ half-plane. Let us assume that the following estimate holds on the real axis as $|u| \to \infty$:

$$\frac{L(u)}{u} = \frac{1}{|u|} [1 + O(e^{-\nu |u|})], \qquad \nu > 0 \tag{1.4}$$

We shall call a kernel $K(t)$ of the form (1.2) with the described function $L(u)u^{-1}$ the "main variant". A number of mixed problems for an elastic strip and an elastic wedge [8 and 9] reduces to the integral equation (1.1) with such a kernel. The theory expounded below is most complete and final for the main variant. In other contact problem cases, when some of the listed properties of the function $L(u)u^{-1}$ are not satisfied, the results expounded below are either valid, or as a rule, can be modified appropriately. We present more details on this in Section 6.

On the basis of the property (1.4), the kernel $K(t)$ can be represented for all $0 \leqslant |t| < \infty$ as [8]

$$K(t) = -\ln|t| + F(t), \qquad F(t) = \int_0^\infty \frac{[L(u) - 1]\cos ut + e^{-u}}{u} du \tag{1.5}$$

where the function $F(t)$ with all its derivatives is continuous in the mentioned range of variation in t.

Let us assume that the function $f(x)$ in the right side of (1.1) belongs to the class $H_1^\alpha(-1,1)$ (see [8], say), then on the basis of (1.5) the integral equation (1.1) can be represented as an equivalent integral equation of the second kind in $L(-1,1)$ for $\lambda > 0$ [8]

$$\varphi(x) = \varphi_0(x) + \frac{1}{\pi^2 \lambda \sqrt{1 - x^2}} \int_{-1}^{1} \frac{\sqrt{1 - \tau^2}}{\tau - x} d\tau \int_{-1}^{1} \varphi(\xi) F'\left(\frac{\tau - \xi}{\lambda}\right) d\xi$$

$$\varphi_0(x) = \frac{1}{\pi\sqrt{1 - x^2}} \left[P - \int_{-1}^{1} \frac{f'(\tau)\sqrt{1 - \tau^2}}{\tau - x} d\tau\right] \tag{1.6}$$

with the additional condition

$$P = \int_{-1}^{1} \varphi(\xi)\, d\xi = \frac{1}{\ln 2\lambda}\left[\int_{-1}^{1} \frac{f(\tau)\, d\tau}{\sqrt{1 - \tau^2}} - \frac{1}{\pi} \int_{-1}^{1} \frac{d\tau}{\sqrt{1 - \tau^2}} \int_{-1}^{1} \varphi(\xi) F\left(\frac{\tau - \xi}{\lambda}\right) d\xi\right] \tag{1.7}$$

On the basis of (1.6), the following two theorems can be proved.

Theorem 1.1. If the solution of the integral equation (1.1) in the class $L(-1,1)$ exists for all $\lambda > 0$, then it has the form

$$\varphi(x) = \varphi_0(x) + \omega(x)(1 - x^2)^{-1/2} \tag{1.8}$$

where the function $\omega(x)$ is continuous with all its derivatives for $x \in [-1,1]$.

In order to formulate the second theorem, let us introduce the norm in the space of functions H_n^{α} $(-\beta,\beta)$ by means of the following relationship:

$$\|f(x)\| = \sum_{m=0}^{n} \max|f^{(m)}(x)| + \sup \frac{|f^{(m)}(t) - f^{(m)}(x)|}{|t-x|^{\alpha}}, \qquad (x, t) \in [-\beta, \beta] \quad (1.9)$$

Theorem 1.2. If solutions of the integral equation (1.1) and

$$\int_{-1}^{1} \varphi_* (\xi) K_* \left(\frac{\xi - x}{\lambda}\right) d\xi = \pi f(x), \qquad |x| \leqslant 1 \quad (1.10)$$

exist and are unique in the class of functions $L(-1,1)$ for $\lambda > \lambda_0$, where

$$\|F(t) - F_*(t)\|_{H_n^{\alpha}(-2/\lambda,\, 2/\lambda)} \leqslant \varepsilon \quad (1.11)$$

then the following estimate is valid:

$$\|\varphi(x) - \varphi_*(x)\|\ C_{n-1(-1,1)} \leqslant \lambda_0^{-1} D\varepsilon (1-x^2)^{-1/2}, \quad D = \text{const} \quad (1.12)$$

By virtue of Theorem 1.2, the following lemma is of interest.

Lemma 1.1. If $K_*(t)$ has the form (1.2) and

$$|L(u)u^{-1} - L^*(u)u^{-1}| < \delta(u^2 + C^2)^{-3/2} \quad \begin{pmatrix}\delta,\ C\text{-const} \\ |u| \in [0, \infty]\end{pmatrix} \quad (1.13)$$

then (1.11) is valid for $n = 1$.

Theorems 1.1, 1.2 and Lemma 1.1 disclose broad possibilities for an approximate solution of the integral equation (1.1).

Henceforth, to simplify elucidation of the material we limit ourselves to the examinaion of the case $f(x) \equiv 1$, particularly since the solution of (1.1) can be found by means of the known solution for $f(x) \equiv 1$ even for the general case [10].

2. Asymptotic solution of the integral equation (1.1) for large values of the parameter λ. Let us note that for large λ the variable $t = (\xi - x)/\lambda \leqslant 2/\lambda$ is small, and we represent $F(t)$ in the form of the following series [8]:

$$F(t) = \sum_{n=0}^{\infty} a_n t^{2n} \quad (2.1)$$

where the coefficients a_n are

$$a_0 = \int_0^{\infty} \frac{L(u) - 1 + e^{-u}}{u} du, \qquad a_n = \frac{(-1)^n}{(2n)!} \int_0^{\infty} [L(u) - 1] u^{2n-1} du \quad (2.2)$$

$$(n = 1, 2, \ldots)$$

On the basis of (1.4) it can easily be shown that for large n the following estimate holds for a_n :

$$|a_n| = O(\nu^{-2n}) \quad (2.3)$$

It hence follows that the series (2.1) converges absolutely for $\lambda > 2/\nu$. This estimate establishes the theoretical limits for using the approximate solutions of (1.1) which can be obtained on the basis of the expansion (2.1).

Let us now seek the solution of the integral equation (1.1) as [8]

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x) \lambda^{-2n} \quad (2.4)$$

Substituting (2.1) and (2.4) into (1.6) and equating members on the right and left sides with identical powers of λ, we obtain

$$\varphi_0(x)=\frac{P}{\pi}(1-x^2)^{-1/2},\qquad \varphi_1(x)=\frac{2a_1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-x}d\tau\int_{-1}^{1}\varphi_0(\xi)(\tau-\xi)\,d\xi$$

$$\varphi_2(x)=\frac{1}{\pi^2\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-\tau^2}}{\tau-x}d\tau\int_{-1}^{1}[\varphi_0(\xi)\,4a_2(\tau-\xi)^3+\varphi_1(\xi)\,2a_1(\tau-\xi)]\,d\xi \qquad \text{etc.} \tag{2.5}$$

The sequential determination of $\varphi_n(x)$ from (2.5) is not difficult, after which the quantity P can be found from the relationship (1.7). Limiting ourselves to terms of order λ^{-4}, we obtain [8]

$$\varphi(x)=\frac{P}{\pi\sqrt{1-x^2}}\left[1+\frac{2a_1}{\lambda^2}\left(\frac{1}{2}-x^2\right)+\frac{4a_2}{\lambda^4}\left(\frac{7}{8}-x^2-x^4\right)+O(\lambda^{-6})\right]$$

$$P=\pi\left[\ln 2\lambda+a_0+\frac{a_1}{\lambda^2}-\frac{a_1^2}{4\lambda^4}+\frac{9a_2}{4\lambda^4}+O(\lambda^{-6})\right]^{-1} \tag{2.6}$$

As a rule, (2.6) yields sufficient accuracy for practice for $\lambda \geqslant 4/\nu$.

The elucidated method has been developed and utilized to solve a number of specific mixed problems of elasticity theory in [8 and 11 – 19].

3. First method of reducing the integral equation (1.1) to an infinite algebraic system. Let us represent the function $F(t)$ in the form (1.5) as the following double series in Chebyshev polynomials [20]:

$$F\left(\frac{\xi-x}{\lambda}\right)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{ij}(\lambda)T_i(\xi)T_j(x) \tag{3.1}$$

We also expand the function $\omega(x)$ in (1.8) in a series of Chebyshev polynomials

$$\omega(x)=\sum_{i=1}^{\infty}S_iT_{2i}(x) \tag{3.2}$$

Because of the properties of the functions $F(t)$ and $\omega(x)$ mentioned above, the series (3.1) and (3.2) converge uniformly, respectively, to $F(t)$ in the set of variables $(\xi, x)\in[-1,1]$ and any values of the parameter $\lambda>0$ and to $\omega(x)$ for all $x\in[-1,1]$.

Now, substituting (1.5), (3.1), (1.8) and (3.2) into the integral equation (1.1), we obtain the following infinite system of linear algebraic equations to determine the coefficients S_i of the series (3.2) after a number of manipulations:

$$x_i=R_i+\sum_{j=1}^{\infty}a_{ij}x_j \qquad (i=1,2,\ldots) \tag{3.3}$$

with additional condition

$$P\pi^{-1}(\ln 2\lambda+c_{00})=1+\sum_{j=1}^{\infty}a_{0j}x_j \tag{3.4}$$

Here

$$x_i=S_i(2i)^{-1},\ a_{ij}=-jc_{2i,2j},\ R_i=-Pc_{2i,0}\,\pi^{-1} \tag{3.5}$$

After having solved the infinite system (3.3), the condition (3.4) aids in determining P.

The coefficients a_{ij} of the infinite system (3.3) can be represented on the basis of (1.5) and (3.1), (3.5) as

$$a_{ij}=(-1)^{i+j+1}j\beta_{ij}\int_0^{\infty}\frac{[L(u)-1]}{u}J_{2i}\left(\frac{u}{\lambda}\right)J_{2j}\left(\frac{u}{\lambda}\right)du \qquad (i+j>0) \tag{3.6}$$

$$\beta_{00}=1,\qquad \beta_{i0}=\beta_{0j}=2,\qquad \beta_{ij}=4$$

The constant $c_{00}(\lambda)$ is given by Expression

$$c_{00}(\lambda)=\int_0^\infty \frac{[L(u)-1]\,J_0^2(u/\lambda)+e^{-u}}{u}\,du \tag{3.7}$$

For large i, j and λ the following estimate can be obtained for the coefficients a_{ij} on the basis of (1.4):

$$a_{ij}=O\left[\frac{p^{2p+1/2}}{(4\lambda)^{2p}\,i^{2i+1/2}\,j^{2j-1/2}}\right],\quad p=i+j \tag{3.8}$$

On the grounds of the fact that $L(u)\sim Au$ for $u\to 0$, we find for the coefficients a_{ij} for small λ

$$a_{ij}\sim 0\quad \text{for}\quad i\neq j,\ a_{ii}\sim 1 \tag{3.9}$$

It can now be proved that the infinite system (3.3) is quasi-completely regular for $\lambda > {}_{1/2\nu}$. For small values of λ it becomes unstable.

Therefore, the elucidated method of solving the integral equation (1.1) by reduction to the infinite system (3.3) will be effective only for sufficiently large values of the parameter λ.

It is convenient to solve the infinite system by the method of reduction, because the truncated system has an almost triangular matrix. Specific examples show that the proposed method of solution is actually effective for sufficiently large λ since the accuracy necessary for practice is achieved in this case even when solving a truncated system consisting of two - four equations.

The expounded method has been developed and utilized for the solution of a number of specific plane mixed problems of elasticity theory in [20 - 23]. However, there are already vestiges in [2, 5 and 8].

4. Asymptotic solution of the integral equation (1.1) for small values of the parameter λ. Besides the properties of the function $L(z)\,z^{-1}$ mentioned at the beginning of Section 1, we shall moreover assume that the following estimate hold in any regular [24] system of contours C_k

$$L(z)z^{-1}=O\,(z^{-1})\qquad \text{for}\quad k\to\infty \tag{4.1}$$

Utilizing (1.3) and (4.1), let us represent the kernel $K(t)$ as the sum of residues at the poles ζ_n

$$K(t)=\sum_{m=1}^{\infty} s_m e^{i\zeta_m t},\qquad s_m=\pi i\left\{\left[\frac{\zeta_m}{L(\zeta_m)}\right]'\right\}^{-1} \tag{4.2}$$

It is hence seen that as $|t|\to\infty$

$$K(t)\sim \exp\,(-\varkappa\,|t|),\quad Re\varkappa=\operatorname{Inf}\,(\operatorname{Re}\gamma_m) \tag{4.3}$$

It usually turns out that $\varkappa=\gamma_1$.

Let us now represent the integral equation (1.1) as an equivalent system of three integral equations

$$\int_{-1}^{\infty}\omega\left(\frac{1+\xi}{\lambda}\right)K\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi+\int_{-\infty}^{-1}\left[\omega\left(\frac{1-\xi}{\lambda}\right)-v(\xi)\right]K\left(\frac{\xi-x}{\lambda}\right)d\xi\quad(-1<x<\infty) \tag{4.4}$$

$$\int_{-\infty}^{1}\omega\left(\frac{1-\xi}{\lambda}\right)K\,\frac{\xi-x}{\lambda}\,d\xi=\pi+\int_{-1}^{\infty}\left[\omega\left(\frac{1+\xi}{\lambda}\right)-v(\xi)\right]K\left(\frac{\xi-x}{\lambda}\right)d\xi\quad(-\infty<x<1)$$

$$\int_{-\infty}^{\infty}v(\xi)\,K\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi\qquad(-\infty<x<\infty) \tag{4.5}$$

under the condition

$$\varphi(\xi)=\omega\left(\frac{1+\xi}{\lambda}\right)+\omega\left(\frac{1-\xi}{\lambda}\right)-v(\xi), \qquad |\xi|\leqslant 1 \tag{4.6}$$

The solution of (4.5) is found easily to be

$$v(\xi) = (A\lambda)^{-1} \tag{4.7}$$

By a change of variable the integral equations (4.4) and (4.5) are reduced to just the one

$$\int_0^\infty \omega(\tau) K(\tau-t)\,d\tau=\frac{\pi}{\lambda}+\int_{2/\lambda}^\infty [\omega(\tau)-(A\lambda)^{-1}]\,K\left(\frac{2}{\lambda}-\tau-t\right)d\tau \quad (0\leqslant t<\infty) \tag{4.8}$$

For small λ the asymptotic solution of (4.8) can be obtained by successive approximations. At each step it is hence necessary to find the solution of the same Wiener-Hopf equation, but with different right sides. The zero approximation is found from Eq.

$$\int_0^\infty \omega_0(\tau)\,K(\tau-t)\,d\tau=\frac{\pi}{\lambda} \qquad (0\leqslant t<\infty) \tag{4.9}$$

Let us obtain the solution $\psi(\tau)$ of an integral equation such as (4.9) but with a more general right side $\pi e^{-\mu t}$. To do this, let us first factorize [25] the function $L(z)z^{-1}$

$$\frac{L(z)}{z}=K_+(z)\,K_-(z), \qquad K_+(z)=\prod_{n=1}^\infty \frac{(z+i\delta_n)}{(z+i\gamma_n)}, \qquad K_-(z)=\prod_{n=1}^\infty \frac{(z-i\delta_n)}{(z-i\gamma_n)} \tag{4.10}$$

It is seen from (4.10) that for regularity strips [25] to exist the complex constant μ should satisfy the inequality $\mathrm{Re}\ \mu<\tau_+=\inf\ (\mathrm{Re}\gamma_n,\ \mathrm{Re}\ \delta_n)$.

On the basis of (4.1) and other properties of $L(z)\,z^{-1}$ it can be shown that the estimate

$$K_+(z) = O(z^{-1/2}) \qquad \text{for}\quad k\to\infty \tag{4.11}$$

holds on any regular [24] system of contours C_k .

If it is assumed that the Fourier trasformant $\Omega_+(z)$ of the function $1/2(1+\mathrm{sgn}\,t)\,\omega_0(t)$ tends to zero as $z\to\infty$, then by omitting traditional computations accompanying the utilization of the Wiener-Hopf method [25], we obtain

$$\psi(t)=\frac{1}{2\pi iK_-(-i\mu)}\int_{-\infty}^\infty \frac{e^{-i\alpha t}d\alpha}{K_+(\alpha)(\alpha+i\mu)}= \qquad \begin{pmatrix}0\leqslant t<\infty\\ \mu\neq\delta_n\end{pmatrix}$$

$$=\frac{e^{-\mu t}(-i\mu)}{L(-i\mu)}+\frac{1}{iK_-(-i\mu)}\sum_{m=1}^\infty \frac{e^{-\delta_m t}}{(\mu-\delta_m)\,K_+'(-i\delta_m)} \tag{4.12}$$

Let us note that by virtue of (4.11) the function $\psi(t)$ has a singularity of the form $t^{-1/2}$ as $t\to 0$ in complete conformity with Theorem 1.1. This singularity is expanded into a series in (4.12).

It follows from (4.12) that for $\mu=\gamma_k$ and $t\to\infty$

$$\psi(t)\sim\exp(-\chi t),\ \ \mathrm{Re}\ \chi=\inf\ (\mathrm{Re}\ \delta_n) \tag{4.13}$$

It usually turns out that $\chi=\delta_1$.

Solving (4.8) by successive approximations, and estimating each approximation on the basis of (4.3) and (4.13) we obtain that the solution of (1.1) for the main variant of the kernel $K(t)$ and $f(x)\equiv 1$ is representable for small λ as

$$\varphi(x)=\Phi(x)+\sum_{n=1}^\infty\left[\omega_n^*\left(\frac{1+x}{\lambda}\right)+\omega_n^*\left(\frac{1-x}{\lambda}\right)\right]\exp\left(-\frac{2n\chi}{\lambda}\right) \tag{4.14}$$

where $\Phi(x)$ is

$$\Phi(x) = \omega_0\left(\frac{1+x}{\lambda}\right) + \omega_0\left(\frac{1-x}{\lambda}\right) - (A\lambda)^{-1} \tag{4.15}$$

The function $\omega_0(t)$ is the solution of (4.9) and the functions $\omega_n^*(t)$ are determined successively from

$$\int_0^\infty \omega_n^*(\tau) K(\tau - t)\, d\tau = e^{2\chi\lambda/\lambda} \int_{2/\lambda}^\infty \omega_{n-1}^*(\tau) K\left(\frac{2}{\lambda} - \tau - t\right) d\tau \tag{4.16}$$

$$\omega_0^*(\tau) = \omega_0(\tau) - (A\lambda)^{-1}, \quad 0 \leqslant t < \infty$$

as $t \to 0$ the functions $\omega_n^*(t)$ have singularities of the form $t^{-1/2}$ as $t \to \infty$ they decrease as $\exp(-\chi t)$ and as $\lambda \to 0$ will behave as $O(1)$ for fixed t.

As specific examples show, for practical purposes it is completely adequate to limit oneself to the first terms of $\Phi(x)$ in (4.14). As a rule this will assure the accuracy needed in practice for all $\lambda \leqslant 4/\nu$.

Let us note that sometimes it is more convenient to use in place of $\Phi(x)$

$$\varphi_0^*(x) = A\lambda\omega_0\left(\frac{1+x}{\lambda}\right)\omega_0\left(\frac{1-x}{\lambda}\right) \tag{4.17}$$

because, as is easy to show, the difference between them is on the order of $\exp(-2\chi/\lambda)$.

The elucidated method is given in [20]. The main term of the asymptotic of the solution $\Phi(x)$ (or $\varphi_0^*(x)$) of (1.1) is obtained for small λ and is used to solve a number of specific plane mixed problems of elasticity theory in [18, 26 – 29]. For small λ the main term of the "symmetric" asymptotic of the solution of (1.1) has been obtained in [30 and 31]. In a number of papers [14, 19, 32 and 33] problems have been investigated for which the function $L(u)u^{-1}$ is different, in its properties, from the main variant described in Section 1, nevertheless the main term of the asymptotic of their solution has successfully been constructed for small λ.

5. Second method of reducing the integral equation (1.1) to an infinite algebraic system. Substituting $K(t)$ in the form (4.2) into the integral equation (4.8), we represent it as

$$\int_0^\infty \omega(\tau) K(\tau - t)\, d\tau = \frac{\pi}{\lambda}\left(1 + \sum_{m=1}^\infty s_m C_m e^{-t\gamma_m}\right) \tag{5.1}$$

$$C_m = \frac{\lambda e^{2\gamma_m/\lambda}}{\pi} \int_{2/\lambda}^\infty \left[\omega(\tau) - (A\lambda)^{-1}\right] e^{-\tau\gamma_m}\, d\tau \tag{5.2}$$

Inverting the right side of (5.1) on the basis of (4.12), we obtain

$$\omega(t) = \frac{1}{A\lambda} + \frac{1}{i\lambda}\sum_{k=1}^\infty \frac{e^{-\delta_k t}}{K'_+(-i\delta_k)}\left[-\frac{1}{K_-(0)\delta_k} + \sum_{n=1}^\infty \frac{s_n C_n}{K_-(-i\gamma_n)(\gamma_n - \delta_k)}\right] \tag{5.3}$$

Substituting (5.3) into (5.2) and integrating, we arrive at an infinite algebraic system in C_m of the form

$$X = A_* X + B \tag{5.4}$$

It can be proved [34] that the system (5.4) is completely regular for sufficiently small λ and completely quasi-regular for all $\lambda < \infty$.

It is clear that the method elucidated is effective for sufficiently small λ. It is convenient to solve the system (5.4) by successive approximations. As a rule, it is possible to limit oneself to the zero approximation with sufficient accuracy for practice, namely, to $C_m = 0$.

The expounded method has been developed and utilized to solve a number of specific plane mixed problems of elasticity theory in [34 and 35], but there are already vestiges in [25, 36 and 37].

In conclusion, let us note that the methods of Sections 2 - 5 as a whole permit, as a rule, a complete investigation of any kind of mixed problem, and the representation of its solution in a form suitable for practical utilization.

6. Some generalizations. Example. Three-dimensional (axisymmetric and nonaxisymmetric) nonclassical contact problems. As has been noted above, in order to find the main part of the asymptotic of the solution of integral equation (1.1) for small λ it is necessary to know the solution of the Wiener-Hopf integral equation (4.9). A solution in the form (4.12) is of no practical use since it is represented by a series and does not explicitly contain the singularity $t^{-1/2}$ in the neighborhood of $t = 0$. To obtain practically acceptable solutions of (4.9), the idea of approximate factorization [38] can be utilized.

For example, let us approximate the function $L(u)u^{-1}$, described in Section 1 by the following expression

$$\frac{L^*(u)}{u} = \frac{\sqrt{u^2+B^2}}{u^2+c^2}\prod_{n=1}^{N}\frac{(u^2+D_n^2)}{(u^2+E_n^2)}, \qquad A = \frac{B}{C^2}\prod_{n=1}^{N}\frac{D_n^2}{E_n^2} \tag{6.1}$$

where the numbers B, C, D_n, E_n are all distinct, real and positive. The solution of (4.9) for the case (6.1) is found easily enough and explicitly contains the afore-mentioned singularity. It should be noted that a whole series of nonclassical contact problems of elasticity theory results also in the integral equation (1.1) with a kernel of (1.2) and (6.1) type. Hence, the case (1.2) and (6.1) merits special attention.

Moreover, let us consider the least complex variant for simplicity

$$L^*(u)u^{-1} = (u^2+1)^{-1/2} \tag{6.2}$$

however, let us note that it reflects the nature of the general case (6.1) quite completely.

The kernel $K(t)$ for the variant (6.2) is the Macdonald function $K_0(t)$.

The problem of impressing an infinite bar die with a base $\delta(x, y) = \delta e^{i\beta y}$ on an elastic half-space reduces to an integral equation with such a kernel. Here $\lambda = (a\beta)^{-1}$, $f(x) = \delta G(1-\nu)^{-1}a^{-1}$, a is the half-width of the die, G the shear modulus, ν the Poisson coefficient. The contact pressures are given by

$$q(x,\ y) = \varphi(x)e^{i\beta y} \tag{6.3}$$

Moreover, still another dynamic mixed problem results in the integral equation (1.1) with the kernel $K_0(t)$ Namely, let an infinite undeformed strip of width $2a$ be clamped rigidly to the surface of an elastic half-space, where a tangential force $T = T_0 e^{\varkappa t}$, t is the time, acts on each unit length of the strip. Under the effect of these forces the strip is shifted by a quantity $\gamma(t) = \gamma_0 e^{\varkappa t}$. The distribution function of the tangential stresses is

$$\tau(x,\ \ t) = \varphi(x)\, e^{\varkappa t} \tag{6.4}$$

where $\varphi(x)$ is the solution of (1.1). Here

$$\lambda = (a\varkappa)^{-1}\rho^{-1/2}\, G^{1/2},\ f(x) = G\gamma_0 a^{-1},$$

where ρ is the density of the elastic half-space.

The integral equation (1.1) with kernel $K_0(t)$ has been studied earlier in [39 and 40]. The methods elucidated above are used for its solution here.

The asymptotic solution of the form (2.6) for large λ is not applicable in the considered case since the asymptotic of the function $F(t)$ is, in contrast to (2.1), for small t

$$F(t)=\sum_{i=0}^{\infty} t^{2i}(a_i+b_i\ln|t|) \tag{6.5}$$

However, modifying the scheme expounded in Section 2 somewhat, namely, seeking the solution $\varphi(x)$ of the integral equation (1.6) as a double series in powers of λ^{-2} and in $\ln\lambda$, we obtain

$$\begin{gathered}\varphi(x)=\frac{P}{\pi\sqrt{1-x^2}}\left[1+\left(a_1+\frac{3}{2}b_1-b_1\ln 2\lambda\right)(1-2x^2)\lambda^{-2}+O(\lambda^{-4}\ln^2 2\lambda)\right]\\ P=\pi f(0)\left[a_0+\ln 2\lambda+(a_1+b_1-b_1\ln 2\lambda)\lambda^{-2}+O(\lambda^{-4}\ln^2 2\lambda)\right]^{-1}\end{gathered} \tag{6.6}$$

For the considered case of $K(t)=K_0(t)$, the constants in (6.6) are

$$a_0=0{,}1159,\quad a_1=0.2790,\quad b_1=-0.2500 \tag{6.7}$$

Let us note that the power-logarithmic asymptotic of the solution of nonclassical plane contact problems of the form (6.6) for large λ has first been obtained and used to study specific problems in [41] and [42].

The main term of the asymptotic of the solution of the considered problem for small is obtained by means of (4.15) by first solving the Wiener-Hopf integral equation (4.9) for the variant (6.2). Let us present the final result

$$\omega_0(t)=\frac{f(0)}{\lambda}\left(\operatorname{erf}\sqrt{t}+\frac{1}{\sqrt{\pi t}}e^{-t}\right)\ (\operatorname{erf} x \text{ is the probability integral}) \tag{6.8}$$

As can be shown on the basis of (4.8), a more exact solution of the integral equation (1.1) for the considered problem with small λ adds terms on the order of $\exp(-2/\lambda)$ and higher to (6.8).

On the basis of (6.8) we obtain for P

$$P=\int_{-1}^{1}\varphi(\xi)\,d\xi=f(0)\left[(2s+1)\operatorname{erf}\sqrt{s}-s+2\sqrt{\frac{s}{\pi}}e^{-s}\right],\qquad s=\frac{2}{\lambda} \tag{6.9}$$

Some results of calculations are given below for $\lambda=2$ by using (6.6)(upper values) and (4.15),(6.8) and (6.9) (lower values)

$$\frac{\varphi(0)}{f(0)}=\begin{cases}0.669,\\0.667,\end{cases}\qquad \frac{\psi}{f(0)}=\begin{cases}0.411,\\0.407,\end{cases}\qquad \frac{P}{f(0)}=\begin{cases}1.96\\1.94\end{cases}$$

The quantity ψ is given by the relationship

$$\psi=\lim\ \varphi(\xi)(1+\xi)^{1/2},\quad \xi\to 1 \tag{6.10}$$

It is hence seen that the asymptotic solutions for large and small λ merge with a sufficient degree of accuracy for practical purposes.

It should be noted that the first method of infinite algebraic systems remains valid for the case (6.1), however, the second method of systems cannot successfully be extended to the case (6.1).

In conclusion, let us briefly consider three-dimensional nonclassical contact problems. Nonclassical contact problems for an elastic half-space and a layer with a circular zone of separation of the boundary conditions here comprise a large group. Such problems may, as a rule, be reduced to the solution of the following integral equation of the first kind with a symmetric nonregular kernel

$$\int_0^1 \varphi_n(\rho)\rho K_n(\rho/\lambda,\ r/\lambda)\,d\rho = \lambda f_n(r) \qquad (0 \leqslant r \leqslant 1)$$
$$K_n(\mu,\ \nu) = \int_0^\infty L(u)\,J_n(u\mu)\,J_n(u\nu)\,du \tag{6.11}$$

where $L_n(x)$ are Bessel functions, and the function $L(u)$ has the form (1.3), (1.4) or (6.1).

All the methods of approximately solving (1.1) described in Sections 2 – 5 may be applied, with appropriate modifications, to study the integral equation (6.11). Namely, an asymptotic of (6.11) for large λ has been obtained and used in [6, 13, 20 and 43 – 53] to solve specific nonclassical contact problems.

The first method of infinite algebraic systems has been developed and used in [20 and 54] to solve the integral equation (6.11).

The main term of the asymptotic of the solution of (6.11) for small λ has been obtained in [20 and 31].

The second method of infinite algebraic systems has been developed for (6.11) in [35].

If the zone of separation of the boundary conditions on the surface of a half-space of a layer is not circular, then the corresponding nonclassical contact problems result in the solution of the following integral equation of the first kind with a nonregular difference kernel:

$$\iint_\Omega \varphi(P)\,K\left(\frac{R_{PQ}}{\lambda}\right)dP = 2\pi f(Q) \quad (Q \in \Omega), \quad K(t) = \int_0^\infty L(u)\,J_0(ut)\,u u \tag{6.12}$$

where Ω is the domain of contact, R_{PQ} the distance between the points P and Q. The function $L(u)$ has the form (1.3), (1.4) or (6.1).

Only the methods elucidated in Sections 2 and 4 have already been used successfully, with appropriate modifications, for the approximate solution of (6.12). Namely, an asympotic solution of (6.12) for large λ has been obtained and used in [7, 55 and 56] to solve specific problems. The main term of the asymptotic of the solution of (6.12) for small λ has been constructed in [57]. Apparently even transfer of the first method of infinite algebraic systems to the case (6.12) will raise no obstacles.

Finally, let us note that the methods of Sections 2 – 4 can successfully be utilized also to investigate the integral equation

$$\int_a^b \varphi_n(\rho)\,\rho d\rho \int_0^\infty J_n(u\rho)\,J_n(ur)\,du = f_n(r) \qquad (a \leqslant r \leqslant b) \tag{6.13}$$

to which mixed problems for an elastic half-space with an annular zone of separation of the boundary conditions will reduce. Namely, an asymptotic of the solution of (6.13) has been obtained in [41] for large λ, and asymptotic solutions of (6.13) for small λ (large $\varepsilon = a/b$) have been obtained in [41, 58 – 61] by using methods analogous to those expounded in Section 4.

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